Localization in quantum field theory

Fumio Hiroshima*
Faculty of Mathematics, Kyushu University

Abstract

This is a brief review of the paper [14]. The Nelson Hamiltonian can be realized as a self-adjoint operator defined on the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathcal{F}$. It is shown that the strictly positive ground state $\varphi_{\rm g}$ of the Nelson Hamiltonian

$$(-\frac{1}{2}\Delta + V) \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + \phi$$

with external potential $V(x) = |x|^{2n}$ satisfies that

$$C_1 e^{-C_2|x|^{n+1}} \leq \|\varphi_{\rm g}(x)\|_{\mathcal{F}} \leq C_3 e^{-C_4|x|^{n+1}} \quad a.e.\ x \in \mathbb{R}^d$$

with some constants C_1, C_2, C_3 and C_4. The upper bound mentioned above is shown in e.g. [15, Section 2.9.1]. The main contribution of this article is to give the lower bound.

1 Heuristic discussions

In general significant localizations in quantum field theory include boson number localizations, Gaussian dominations on ϕ and spatial localizations on x. Let Ψ be a bound state of a Hamiltonian H, i.e., $H\Psi = E\Psi$. Ψ is a function of field variable ϕ, spatial variable x and the number of bosons n. Hence it should be written as $\Psi = \Psi(x, \phi, n)$. We expect that

$$\Psi(x, \phi, n) \sim e^{-\phi^2}, \quad \Psi(x, \phi, n) \sim e^{-|x|^m}, \quad \Psi(x, \phi, n) \sim e^{-\beta n}.$$

The precise meaning of these are given in e.g., [11, Section 5] for the localization on x, and [9, Sections 4.2.1 and 4.4.1] for those of ϕ and n. In this article we are concerned with localizations on spatial variable x.

*e-mail: hiroshima@math.kyushu-u.ac.jp

We begin with a heuristic discussion of the spatial decay of bound states of Schrödinger operators. It is well known that bound states of the harmonic oscillator

$$h_{\rm osc} = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$$

are of the form $f_n(x) = H_n(x)e^{-|x|^2/2}$ with the nth-degree Hermite polynomial $H_n(x)$. Hence

$$f(x) \sim e^{-|x|^2}$$

as $|x| \to \infty$. Let us consider more general cases. We consider the one-dimensional Schrödinger operator of the form

$$h = -\frac{1}{2}\Delta + |x|^{2n}.$$

Let f be a bound state of h such that

$$hf = Ef.$$

Substituting $f(x) = e^{-|x|^m}$ in the above identity, we see that

$$-\frac{1}{2}(m(m-1)|x|^{m-2} + m^2|x|^{2m-2})f(x) + |x|^{2n}f = Ef.$$

By the dominant part of the left-hand side, $-\frac{1}{2}m^2|x|^{2m-2} + |x|^{2n}$, we conclude that $m = n + 1$ and

$$f(x) \sim e^{-|x|^{n+1}}.$$

An alternative heuristic argument used by a path measure is given below. By the identity $f = e^{-t(h-E)}f$ for $t \geq 0$ and the Feynman-Kac formula we see that

$$(g, f) = (g, e^{-t(h-E)}f) = e^{tE}\int_{\mathbb{R}} \bar{g}(x)\mathbb{E}[e^{-\int_0^t V(B_s+x)ds}f(B_t + x)]dx, \qquad (1.1)$$

where $(B_t)_{t\geq 0}$ is Brownian motion on a probability space $(\mathscr{X}, \mathcal{B}, \mathcal{W})$ and $\mathbb{E}[\ldots] = \int_{\mathscr{X}} \ldots d\mathcal{W}$ denotes the expectation. Then we have

$$f(x) = e^{tE}\mathbb{E}[e^{-\int_0^t V(B_s+x)ds}f(B_t + x)] \qquad (1.2)$$

for any $t \geq 0$. We define the subset of $\mathscr{X}$: $A = \{w \in \mathscr{X} \mid \sup_{0\leq s\leq t}|B_s(w)| > a\}$. Hence

$$W_a(x) = \inf\{V(y + x) \mid |y| < a\} \leq \inf_{0\leq s\leq t} V(B_s(w) + x)$$

for $w \in A^c$. The crucial point is that $W_a(x)$ is a deterministic function. Dividing the expectation on the right-hand side of (1.2) as $\mathbb{E}[\ldots] = \mathbb{E}[\mathbb{1}_A \ldots] + \mathbb{E}[\mathbb{1}_{A^c} \ldots]$ we have

$$f(x) \le e^{tE}\|f\|_\infty(\mathbb{E}[\mathbb{1}_A] + \mathbb{E}[\mathbb{1}_{A^c}e^{-tW_a(x)}]).$$

Set $a = |x|/2$. Roughly speaking we see that by Leṽy's maximal inequality

$$\mathbb{E}[\mathbb{1}_A] \le \frac{2}{\sqrt{2\pi t}} \int_{|x+y|>|x|/2} e^{-|y|^2/2t} dy \sim e^{-|x|^2/t}, \tag{1.3}$$

and

$$\mathbb{E}[\mathbb{1}_{A^c}e^{-tW_{|x|/2}(x)}] = \mathbb{E}[\mathbb{1}_{A^c}]e^{-tW_{|x|/2}(x)} \le e^{-tW_{|x|/2}(x)} \sim e^{-t|x|^{2n}}. \tag{1.4}$$

Setting $t = |x|^\alpha$ and comparing (1.3) and (1.4) we see that $2 - \alpha = 2n + \alpha$. Hence $\alpha = -n + 1$ and we conclude that $f(x) \sim e^{-|x|^{n+1}}$.

These heuristic arguments have been established rigorously in [3] for more general V and dimensions. We extend this to a model in QFT in this article.

2 Nelson model with Kato-class potentials

2.1 Definition

Although in [14] general cases are investigated, only a special case is discussed in this article for the readers convenient. The Nelson model [17] describes a linear interaction between non-relativistic spinless nucleons and scalar mesons, and the non-relativistic particles are governed by a Schrödinger operator. Without going into physical interpretation we mainly discuss a mathematical technique in this article. Let

$$H_{\mathrm{N}} = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \phi(\tilde{\varphi}(\cdot - x))$$

be the Nelson Hamiltonian with ultraviolet cutoff function $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})^\vee \in L^2(\mathbb{R}^d)$.[1] H_{N} is defined on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}.$$

[1] $\hat{f}$ denotes the Fourier transform of $f \in L^2(\mathbb{R}^d)$ defined by $\hat{f}(k) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-ikx}dx$, and $\check{f}$ the inverse Fourier transform of f.

Here $\mathcal{F} = L^2(Q)$ be the L^2-space over a probability space (Q, Σ, μ). $\phi(f)$ is the Gaussian random variable indexed by real-valued function $f \in L^2(\mathbb{R}^d)$ on (Q, Σ, μ) such that

$$\int_Q \phi(f) d\mu = 0, \quad \int_Q \phi(f)\phi(g) d\mu = \frac{1}{2}(f, g)_{L^2(\mathbb{R}^d)}.$$

$H_{\rm f} = d\Gamma(\hat{\omega})$ is the free field Hamiltonian defined by the differential second quantization of $\hat{\omega} = \omega(-i\nabla)$ in $L^2(\mathbb{R}^d)$. Here $\omega(k) = \sqrt{|k|^2 + \nu^2}$ with $\nu \geq 0$. Finally

$$H_{\rm p} = -\frac{1}{2}\Delta + V$$

denotes the d-dimensional Schrödinger operator with external potential V. We assume that V is a Kato-decomposable potential defined below.

(1) $V : \mathbb{R}^d \to \mathbb{R}$ is a Kato-class potential whenever

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B_r(x)} |g(x-y)V(y)| \, dy = 0$$

holds, where $B_r(x)$ is the closed ball of radius r centered at x, and

$$g(x) = \begin{cases} |x|, & d = 1, \\ -\log|x|, & d = 2, \\ |x|^{2-d}, & d \geq 3. \end{cases}$$

We denote this linear space by $\mathcal{K}(\mathbb{R}^d)$.

(2) $V \in \mathcal{K}_{\rm loc}(\mathbb{R}^d)$ if and only if $\mathbb{1}_K V \in \mathcal{K}(\mathbb{R}^d)$ for a compact set $K \subset \mathbb{R}^d$.

(3) V is Kato-decomposable whenever $V(x) = V_+(x) - V_-(x)$ with $V_\pm(x) \geq 0$ for $x \in \mathbb{R}^d$, and $V_+ \in \mathcal{K}_{\rm loc}(\mathbb{R}^d)$ and $V_- \in \mathcal{K}(\mathbb{R}^d)$.

Throughout we assume that $\hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(k)$. Hence we see that $H_{\rm N}$ is symmetric and ϕ is relatively bounded with respect to $-\frac{1}{2}\Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f}$.

Remark 2.1 In the Fock representation $\phi(\tilde{\varphi}(\cdot - x))$ and $H_{\rm f}$ are given by

$$\phi(\tilde{\varphi}(\cdot - x)) = \frac{1}{\sqrt{2}} \int \left(a^\dagger(k) e^{-ikx} \frac{\hat{\varphi}(k)}{\omega(k)} + a(k) e^{ikx} \frac{\hat{\varphi}(k)}{\omega(k)} \right) dk,$$

$$H_{\rm f} = \int \omega(k) a^\dagger(k) a(k) dk.$$

Here $a^\dagger(k)$ and $a(k)$ denote the creation operator and the annihilation operator satisfying $[a^\dagger(k), a(k)] = \delta(k - k')$.

The ultraviolet cutoff $\hat{\varphi}$ of the Nelson model can be renormalized. This has been done in [17, 18, 7, 6]. The existence of the ground state of the Nelson Hamiltonian with/without cutoffs is established. See [1, 5, 19] for one with cutoff and [10, 16] for without cutoff. We also refer to [12] for a comprehensive summary on ground states of related models. We discuss the Nelson Hamiltonian with cutoffs in this article.

2.2 Feynman-Kac type formula

Let (Q_E, Σ_E, μ_E) be a probability space, and $\phi_E(f)$ Gaussian random variables on it with $f \in L^2(\mathbb{R}^{d+1})$. It satisfies that

$$\int_{Q_E} \phi_E(f) d\mu_E = 0, \quad \int_{Q_E} \phi_E(f)\phi_E(g) d\mu_E = \frac{1}{2}(f, g)_{L^2(\mathbb{R}^{d+1})}.$$

Let $\mathcal{F}_E = L^2(Q_E)$. To construct a functional integral representation of $(F, e^{-tH_\mathrm{N}}G)$ we introduce isometries J_t connecting $\mathcal{F}$ and the Euclidean field $\mathcal{F}_E$. Let

$$\mathrm{j}_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1})$$

be the family of isometries and

$$\mathrm{J}_t = \Gamma(j_t) : \mathcal{F} \to \mathcal{F}_E$$

the second quantization of j_t. J_t is an isometry from $\mathcal{F}$ to $\mathcal{F}_E$. It satisfies that

$$\mathrm{j}_t^* \mathrm{j}_s = e^{-|t-s|\hat{\omega}}$$

and

$$\mathrm{J}_t^* \mathrm{J}_s = e^{-|t-s|H_\mathrm{f}}$$

for any $t, s \in \mathbb{R}$. We identify $\mathcal{H}$ with $L^2(\mathbb{R}^d \times Q, dx \otimes d\mu)$. See Figure 1.

Let $(B_t)_{t\geq 0}$ be d-dimensional Brownian motion starting at $x \in \mathbb{R}^d$ on a probability space $(\mathscr{X}, \mathcal{B}, \mathcal{W}^x)$ and we write $\mathbb{E}^x[\ldots] = \mathbb{E}_{\mathcal{W}^x}[\ldots] = \int_{\mathscr{X}} \ldots d\mathcal{W}^x$. It is immediate to see that

$$(F, e^{-t(H_\mathrm{p}\otimes \mathbb{1} + \mathbb{1}\otimes H_\mathrm{f})}G) = \int_{\mathbb{R}^d} \mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(F(B_0), \mathrm{J}_0^* \mathrm{J}_t G(B_t))_\mathcal{F}]dx.$$

We give a remark on notations above. $(F(B_0), \mathrm{J}_0^* \mathrm{J}_t G(B_t))_\mathcal{F}$ denotes the inner product of $F(B_0)$ and $\mathrm{J}_0^* \mathrm{J}_t G(B_t)$. Here $F, G \in \mathcal{H}$ and $F(B_0(w)), G(B_t(w)) \in \mathcal{F}$ for each

$w \in \mathscr{X}$ under the identification $\mathcal{H} \cong L^2(\mathbb{R}^d;\mathcal{F})$. I.e., $F = F(\cdot) \in \mathcal{H}$ is an $\mathcal{F}$-valued L^2- function such that $\|F\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^d} \|F(x)\|_{\mathcal{F}}^2 dx$. We set

$$\mathrm{I}_{(0,t)} = \mathrm{J}_0^* e^{-\phi_E(\int_0^t \mathrm{j}_s \tilde{\varphi}(\cdot - B_s)ds)} \mathrm{J}_t. \tag{2.1}$$

Note that $\mathrm{I}_{(0,t)} : \mathcal{F} \to \mathcal{F}$, but it is not clear if $\mathrm{I}_{(0,t)}$ is bounded. We shall show that $\mathrm{I}_{(0,t)}$ is bounded. The following estimate is established in [13]

$$\|\mathrm{I}_{(0,t)}\| \le \sqrt{2} \exp\left\{ \frac{t}{2} \|\hat{\varphi}/\omega\|^2 + 2t(t \vee 1)(\|\hat{\varphi}/\sqrt{\omega}\|^2 + \|\hat{\varphi}/\omega\|^2) \right\}. \tag{2.2}$$

Under the extra assumption $\|\hat{\varphi}/\omega^{3/2}\| < \infty$ it holds that

$$\|\mathrm{I}_{(0,t)}\| \le \sqrt{2} \exp(tE(\hat{\varphi})), \tag{2.3}$$

where

$$E(\hat{\varphi}) = \max\left\{ \frac{3}{2}\|\hat{\varphi}/\omega\|^2 + \|\hat{\varphi}/\omega^{3/2}\|^2, \frac{3}{2}\|\hat{\varphi}/\omega\|^2 + \|\hat{\varphi}/\omega^{1/2}\|^2 \right\}.$$

In particular we have

$$|(\Psi, \mathrm{I}_{(0,t)}\Phi)_{\mathcal{F}}| \le C_{\hat{\varphi}} \|\Psi\|_{\mathcal{F}} \|\Phi\|_{\mathcal{F}},$$

where $C_{\hat{\varphi}}$ is given by the right-hand side of (2.2) or (2.3).

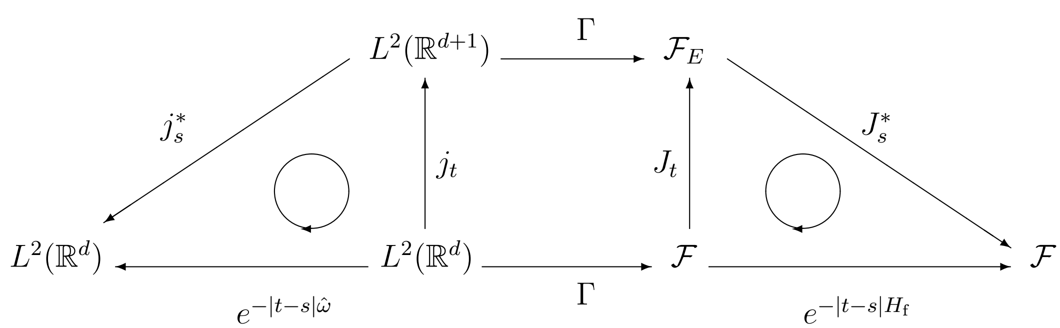

Figure 1: Second quantization Γ, Euclidean field $\mathcal{F}_E$ and isometries J_t

Proposition 2.2 *Let V be Kato-decomposable. Then there exists the unique strongly continuous one-parameter semi-group S_t such that*

$$(F, S_t G)_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathbb{E}^x[e^{-\int_0^t V(B_s)ds} (F(B_0), \mathrm{I}_{(0,t)} G(B_t))_{\mathcal{F}}] dx. \tag{2.4}$$

By the version of Stone's theorem for semi-group there exists the self-adjoint operator bounded from below $H_{\rm N}$ such that

$$S_t = e^{-tH_{\rm N}}, \quad t \geq 0. \tag{2.5}$$

The self-adjoint Nelson Hamiltonian $H_{\rm N}$ with Kato-decomposable potential V is defined by (2.5).

Remark 2.3 If V satisfies that

$$\|Vf\| \leq a\| - \frac{1}{2}\Delta f\| + b\|f\|$$

with $a < 1$, then $H_{\rm N}$ is self-adjoint on $D(-\Delta \otimes 1\!\!1) \cap D(1\!\!1 \otimes H_{\rm f})$. This is due to Kato-Rellich theorem. To define $H_{\rm N}$ with Kato-decomposable V it is standard to use Feynman-Kac type formula. See [8].

3 Spatial decay of the ground state

3.1 Positivity and spatial continuity of the ground state

Let $\varphi_{\rm g}$ be the ground state of $H_{\rm N}$ such that

$$H_{\rm N}\varphi_{\rm g} = E\varphi_{\rm g},$$

where E is the infimum of the spectrum of $H_{\rm N}$.

Lemma 3.1 *Let $\Psi \in \mathcal{F}$. Then*

$$(\Psi, \varphi_{\rm g}(x))_{\mathcal{F}} = e^{tE}\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\Psi, {\rm I}_{(0,t)}\varphi_{\rm g}(B_t))_{\mathcal{F}}] \quad a.e. x \in \mathbb{R}^d. \tag{3.1}$$

Proof: From the identity $\varphi_{\rm g} = e^{-t(H_{\rm N}-E)}\varphi_{\rm g}$ we have

$$(f \otimes \Psi, \varphi_{\rm g})_{\mathcal{H}} = e^{tE}\int_{\mathbb{R}^d} \bar{f}(x)\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\Psi, {\rm I}_{(0,t)}\varphi_{\rm g}(B_t))_{\mathcal{F}}]dx$$

for any $f \in C_0^\infty(\mathbb{R}^d)$. Then we have (3.1). □

Lemma 3.2 *There exist constants $C > 0$ and $b > 0$ such that*

$$\|\varphi_{\rm g}(x)\|_{\mathcal{F}} \leq t^{-d/4}CC_{\hat{\varphi}}(t)e^{bt}\|\varphi_{\rm g}\| \quad a.e. x$$

for any $t > 0$. In particular $\|\varphi_{\rm g}(\cdot)\|_{\mathcal{F}} \in L^\infty(\mathbb{R}^d)$ and there exists $C > 0$ independent of $w \in \mathscr{X}$ and $x \in \mathbb{R}^d$ such that $\|{\rm I}_{(0,t)}\varphi_{\rm g}(B_t)\| \leq C$.

Proof: By $\varphi_{\rm g}(x) = e^{tE}\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}\mathrm{I}_{(0,t)}\varphi_{\rm g}(B_t)]$ we have

$$\|\varphi_{\rm g}(x)\|_{\mathcal{F}} \le e^{tE}C_{\hat{\varphi}}(t)\left(\mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}]\right)^{1/2}\left(\mathbb{E}^x[\|\varphi_{\rm g}(B_t)\|^2]\right)^{1/2}.$$

Since $\mathbb{E}^x[\|\varphi_{\rm g}(B_t)\|^2] = (2\pi t)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/2t}\|\varphi_{\rm g}(y)\|^2 dy \le Ct^{-d/2}\|\varphi_{\rm g}\|^2_{\mathcal{F}}$, and Kato - decomposable potential V implies that

$$\mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}] \le \gamma e^{\beta t}$$

with some γ and β, we have the lemma. $\square$

From (2.4) we see that $e^{-tH_{\rm N}}$ for $t>0$ is positivity improving. Then the ground state $\varphi_{\rm g}$ is strictly positive in $L^2(\mathbb{R}^d\times Q)$. Namely $\varphi_{\rm g}(x,\phi)>0$ for $(x,\phi)\in(\mathbb{R}^d\times Q)\backslash N$ with some null set N. Let $\Psi\in\mathcal{F}$ and $t>0$ we define $\rho_\Psi(x)=\rho_\Psi(x,t)$ by

$$\rho_\Psi(x) = e^{tE}\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\Psi,\mathrm{I}_{(0,t)}\varphi_{\rm g}(B_t))_{\mathcal{F}}].$$

We remark that $\varphi_{\rm g}$ is a vector in $L^2(\mathbb{R}^d\times Q)$. Hence it is meaningless to consider $\varphi_{\rm g}(x)$ for every $x\in\mathbb{R}^d$. The next lemma is a key lemma to show the lower bound of $\|\varphi_{\rm g}(x)\|_{\mathcal{F}}$. We mention it without proofs.

Lemma 3.3 ([14]) *Suppose that V is Kato-decomposable. (1) Let $\Psi\in\mathcal{F}$. Then $\rho_\Psi(x)$ is continuous in x. (2) Let $\Psi\in\mathcal{F}$ be non-negative and $\Psi\not\equiv 0$. Then $\rho_\Psi(x)>0$ for all $x\in\mathbb{R}^d$.*

3.2 Lower bound of spatial decay

For simplicity we assume that $0\le V$ is Kato-decomposable and $V(x)\le\gamma|x|^{2n}$ outside a compact set in $\mathbb{R}^d$. For any compact set $K\subset\mathbb{R}^d$ and non-negative $\Psi\in\mathcal{F}$ from Lemma 3.3 there exists ε_K such that $\inf_{x\in K}\rho_\Psi(x)\ge\varepsilon_K$. We prepare some estimate of probabilities of sets of paths. Let

$$P(a,[c,d],t) = \{w\in\mathscr{X} \mid \sup_{0\le s\le t}|B_s(w)|\le a, B_t(w)\in[c,d]\}.$$

It is well known in e.g. [2, p.174, 1.15.8] that

$$\mathcal{W}(P(a,[c,d],t)) = \frac{1}{\sqrt{2\pi t}}\int_{[c,d]}\sum_{k=-\infty}^{\infty}(-1)^k\exp\left(-\frac{(u-2ka)^2}{2t}\right)du. \qquad (3.2)$$

Note that

$$\sum_{k=-\infty}^{\infty} \exp\left(-\frac{(u-2ka)^2}{2t}\right) < \infty$$

for any u, and

$$\int_{[c,d]} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(u-2ka)^2}{2t}\right) du < \infty.$$

Proposition 3.4 ([3]) *Suppose that $a > 0$, $\alpha > 0$ and $t > 0$ satisfy that $\alpha < a/2$ and $a^2/t > \beta$. Here β is the unique positive solution ξ of the equation*

$$e^{-\xi/2} - (e^{-25\xi/8} + e^{-9\xi/8} + e^{-169\xi/8}) = 0.$$

Then for any $x \in [-(a-\alpha), a-\alpha]$

$$\mathcal{W}(P(a, [x-\alpha, x+\alpha], t)) \geq \frac{\alpha}{\sqrt{2\pi t}} \xi(\frac{a^2}{t}) e^{-\frac{a^2}{2t}}.$$

Proof: This is a minor modification of [3, Lemma 2.1]. $x \in [-(a-\alpha), a-\alpha]$ is equivalent to $[x-\alpha, x+\alpha] \subset [-a, a]$. Let

$$g(x) = \mathcal{W}(P(a, [x-\alpha, x+\alpha], t) = \frac{1}{\sqrt{2\pi t}} \int_{x-\alpha}^{x+\alpha} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2ka)^2}{2t}\right) du.$$

We can see that $g(x) = g(-x)$. We suppose that $x \in [0, a-\alpha]$. Let

$$f(x) = \mathcal{W}(P(a, [x, x+\alpha], t)) = \frac{1}{\sqrt{2\pi t}} \int_{x}^{x+\alpha} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2ka)^2}{2t}\right) du.$$

We have $g(x) \geq f(x)$. Since $f'(x) \leq 0$, f is a monotonously decreasing function and $f(x) \geq f(a-\alpha) \geq \frac{1}{\sqrt{2\pi t}} \int_{a-\alpha}^{a} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2ka)^2}{2t}\right) du$. We have

$$\int_{a-\alpha}^{a} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2ka)^2}{2t}\right) du \geq \alpha \left(e^{-\frac{a^2}{2t}} + f_e(a,t) - f_o(a,t)\right),$$

where

$$f_e(a,t) = \sum_{m \neq 0, m \in \mathbb{Z}} f_e(m) = \sum_{m \neq 0, m \in \mathbb{Z}} \exp\left(-\frac{|(2-8m)a|^2}{8t}\right),$$

$$f_o(a,t) = \sum_{m \in \mathbb{Z}} f_o(m) = \sum_{m \in \mathbb{Z}} \exp\left(-\frac{|(5-8m)a|^2}{8t}\right).$$

We see that $f_e(-m) - f_o(m+1) > 0$ for $m \geq 1$, and $f_e(m) - f_o(-m-1) > 0$ for $m \geq 1$. Hence we have

$$\int_{a-\alpha}^{a} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{(u-2ka)^2}{2t}\right) du \geq \alpha \left(e^{-\frac{a^2}{2t}} - f_o(0) - f_o(1) - f_o(-1)\right).$$

Set $\xi = a^2/t$. $e^{-\frac{a^2}{2t}} - f_o(0) - f_o(1) - f_o(-1) = e^{-\xi/2} - (e^{-25\xi/8} + e^{-9\xi/8} + e^{-169\xi/8})$. Then the lemma is proven. □

Lemma 3.5 *Let $K \subset \mathbb{R}^d$ be compact. Then there exists $\varepsilon_K > 0$ such that*

$$\inf_{B_t+x\in K} (1\!\!1, \mathrm{I}_{(0,t)}\varphi_{\mathrm{g}}(B_t+x)) \geq \varepsilon_K \exp\left(-\frac{1}{\varepsilon_K}\left(\int_0^t ds \int_0^t W(s-r, B_s - B_r)dr\right)^{1/2} M_\infty\right).$$

Here $M_\infty = \sup_{x\in\mathbb{R}^d} \|\varphi_{\mathrm{g}}(x)\|_{\mathcal{F}}$.

Proof: Let $P(A) = \frac{1}{N}\int_A \mathrm{J}_t\varphi_{\mathrm{g}}(B_t)d\mu_E$, $A \in \Sigma_E$, be a probability measure on (Q_E, Σ_E), where $N = \int_{Q_E} \mathrm{J}_t\varphi_{\mathrm{g}}(B_t)d\mu_E$ is the normalizing constant. By Jensen's inequality we have

$$\begin{aligned}(1\!\!1, \mathrm{I}_{(0,t)}\varphi_{\mathrm{g}}(B_t))_{\mathcal{F}} &= N\frac{(1\!\!1, e^{-\phi_E(\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)ds)}\mathrm{J}_t\varphi_{\mathrm{g}}(B_t))_{\mathcal{F}_E}}{N} \\ &\geq Ne^{-(\phi_E(\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot-B_s)ds), \mathrm{J}_t\varphi_{\mathrm{g}}(B_t))/N}.\end{aligned}$$

We have

$$\left|\left(\phi_E\left(\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)ds\right), \varphi_{\mathrm{g}}(B_t)\right)\right| \leq \left(\int_0^t ds \int_0^t W(s-r, B_s - B_r)dr\right)^{1/2} M_\infty,$$

where

$$W(t, X) = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-|t|\omega(k)} e^{-ikX} dk$$

and we used $\inf_{B_t+x\in K}(1\!\!1, \mathrm{J}_t\varphi_{\mathrm{g}}(B_t+x))_{\mathcal{F}_E} \geq \varepsilon_K$. Then the proof is complete. □

Lemma 3.6 *Suppose that $a_1 \ldots, a_d$, $\alpha_1, \ldots, \alpha_d$, $b_1 \ldots, b_d$ and t satisfy that $a_j^2/t > \beta$, $a_j/2 > \alpha_j$ and $|[-a_j, a_j] \cap [-x_j - b_j, -x_j + b_j]| > 2\alpha_j$. Then*

$$\rho_{1\!\!1}(x) \geq e^{-tW_a(x)} e^{tE} \varepsilon_K e^{-t\|\hat{\varphi}/\omega\| M_\infty/\varepsilon_K} \prod_{j=1}^d \frac{\alpha_j}{\sqrt{2\pi t}} \xi\left(\frac{a_j^2}{t}\right) e^{-\frac{a_j^2}{2t}},$$

where $W_a(x) = \sup\{V(y) \mid |y_j - x_j| < a_j, j = 1, \ldots, d\}$.

Proof: Let $A = \cap_{j=1}^d P(a_j, [-b_j - x_j, b_j - x_j], t)$. By Proposition 3.4 we see that for $\alpha_j < a_j/2$ and $a_j^2/t > \beta$

$$\mathcal{W}(\cap_{j=1}^d P(a, [k_j, k_j + 2\alpha], t)) \geq \prod_{j=1}^d \frac{\alpha_j}{\sqrt{2\pi t}} \xi(\frac{a_j^2}{t}) e^{-\frac{a_j^2}{2t}}.$$

for any $k_j \in [-a_j, a_j]$. By the assumption there exists $k_j \in [-a_j, a_j]$ such that $[k_j, k_j + 2\alpha] \subset [-b_j - x_j, b_j - x_j]$. We have

$$\mathcal{W}(A) \geq \mathcal{W}(\cap_{j=1}^d P(a, [k_j, k_j + 2\alpha], t)) \geq \prod_{j=1}^d \frac{\alpha_j}{\sqrt{2\pi t}} \xi(\frac{a_j^2}{t}) e^{-\frac{a_j^2}{2t}} \tag{3.3}$$

and

$$\int_0^t ds \int_0^t W(s - r, B_s - B_r) dr \leq t \|\hat{\varphi}/\omega\|$$

uniformly in paths. Notice that $\int_0^t V(B_s(w) + x) ds \leq t W_a(x)$ for any $w \in A$. We see that

$$\begin{aligned} \rho_{\mathbb{1}}(x) &\geq e^{tE} \mathbb{E}[\mathbb{1}_A e^{-\int_0^t V(x+B_s)ds} \varepsilon_K e^{-t\|\hat{\varphi}/\omega\| M_\infty/\varepsilon_K}] \\ &\geq e^{tE} e^{-tW_a(x)} \varepsilon_K e^{-t\|\hat{\varphi}/\omega\| M_\infty/\varepsilon_K} \mathcal{W}(\cap_{j=1}^d P(a, [k_j, k_j + 2\alpha], t)). \end{aligned}$$

□

Theorem 3.7 *Let $V(x) \leq \gamma |x|^{2m}$ outside $K = [-K_1, K_1] \times \cdots \times [-K_d, K_d]$ with $\gamma > 0$ and $m > 1$. Then there exist constants $\delta, D > 0$ such that $\|\varphi_{\rm g}(x)\|_{\mathcal{F}} \geq D e^{-\delta |x|^{m+1}}$ for almost everywhere $x \in \mathbb{R}^d$.*

Proof: Since $\sup_{x \in \mathbb{R}^d} \|\varphi_{\rm g}(x)\|_{\mathcal{F}} < \infty$ it suffices to show for large enough $|x|$. We also have

$$\|\varphi_{\rm g}(x)\|_{\mathcal{F}} \geq \rho_{\mathbb{1}}(x)$$

for almost everywhere $x \in \mathbb{R}^d$. Hence it is sufficient to estimate $\rho_{\mathbb{1}}(x)$ from below. We set $t = |x|^{-(m-1)}$, $a_j = (1 + |x_j|)$, $\alpha_j = 1/2$, and $b_j = 1$ for $j = 1, \ldots, d$. It can be seen that $a_j^2/t = (1 + |x_j|)^2 / |x|^{-(m-1)} > \beta$, $a_j/2 = (1 + |x_j|)/2 > 1/2 = \alpha_j$ and

$$|[-a_j, a_j] \cap [-x_j - b_j, -x_j + b_j]| = |[-|x_j| - 1, |x_j| + 1] \cap [-x_j - 1, -x_j + 1]| > 1 = 2\alpha_j$$

for $x \in K^c$. The assumptions in Lemma 3.4 is satisfied. Note that

$$W_a(x) = \sup\{V(y) \mid |y_j - x_j| \leq 1 + |x_j|, j = 1, \ldots, d\} \leq \gamma c^{2m} |x|^{2m}$$

with some constant c and we see that $tW_a(x) \le C|x|^{m+1}$. We have

$$
\begin{aligned}
&-\log(\rho_{\mathbb{1}}(x)) \\
&\le tW_a(x) - tE + t\frac{\|\hat{\varphi}/\omega\| M_\infty}{\varepsilon_K} - \log \varepsilon_K - \sum_{j=1}^{d} \left(\log \frac{\alpha_j}{\sqrt{2\pi}} - \frac{a_j^2}{2t} + \log(\frac{5}{8}\frac{a_j^2}{t} - \beta) \right) \\
&\le (4c)^{2m}\gamma |x|^{m+1} + \frac{-E + \frac{\|\hat{\varphi}/\omega\| M_\infty}{\varepsilon_K}}{|x|^{m-1}} - \log \varepsilon_K - \log \left(\frac{1}{2\sqrt{2\pi}} \right)^d \\
&\qquad + \frac{9}{8}|x|^{m-1} \sum_{j=1}^{d} (1 + 2|x_j| + |x_j|^2) - \sum_{j=1}^{d} \log(\frac{5}{8}(|x_j|+1)^2 |x|^{(m-1)} - \beta) \\
&\le |x|^{m+1} \left((4c)^{2m}\gamma + \frac{9}{8}d + \frac{9}{8}d|x|^{-2} + \frac{9}{4}d|x|^{-1} \right) + \frac{-E + \frac{\|\hat{\varphi}/\omega\| M_\infty}{\varepsilon_K}}{|x|^{m-1}} + \log(2\sqrt{2\pi}/\varepsilon_K)^d \\
&\le \delta |x|^{m+1} + D,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta &= (4c)^{2m}\gamma + \frac{9}{8}d + \frac{9}{8}d(\sqrt{d}|K|)^{-2} + \frac{9}{4}d(\sqrt{d}|K|)^{-1}, \\
D &= \frac{-E + \frac{\|\hat{\varphi}/\omega\| M_\infty}{\varepsilon_K}}{(\sqrt{d}|K|)^{m-1}} + \log(2\sqrt{2\pi}/\varepsilon_K)^d.
\end{aligned}
$$

From which we obtain $\rho_{\mathbb{1}}(x) \ge e^{-D} e^{-\delta |x|^{m+1}}$. □

4 Concluding remarks

Although to consider decay properties of bound state itself is very interesting, it is also technically significant to consider the spectrum of Hamiltonians appearing in quantum field theory. Several applications of the decay properties of bound states are found in [13] and references therein.

In [14] the Nelson model with not only confining potentials but also decaying potentials is studied. Furthermore the Nelson model with the relativistic kinematic term

$$
(\sqrt{-\Delta + m^2} - m + V) \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + \phi \tag{4.1}
$$

is investigated. The order of the spatial decay differs depending on if massless $m = 0$ or massive $m > 0$. In particular in the case of $V(x) \to 0$ as $|x| \to \infty$ and $m = 0$, it can be show that

$$\|\varphi_{\rm g}(x)\|_{\mathcal F}\leq \frac{C}{1+|x|^{d+1}}.$$

In [11] the spatial decay of the relativistic Pauli-Fierz model in quantum electrodynamics

$$\sqrt{(-i\nabla\otimes\mathbb{1}-A(x))^2+m^2}-m+V\otimes\mathbb{1}+\mathbb{1}\otimes H_{\rm rad} \tag{4.2}$$

is also studied. Here $A(x)$ denotes the quantized radiation field given by

$$A_\mu(x)=\sum_{j=1,2}\int e_\mu(\mu,j)\left(a^\dagger(k,j)e^{-ikx}\frac{\hat\varphi(k)}{\sqrt{|k|}}+a(k,j)e^{ikx}\frac{\hat\varphi(k)}{\sqrt{|k|}}\right)dk$$

and $H_{\rm rad}$ the free field Hamiltonian

$$H_{\rm rad}=\sum_{j=1,2}\int |k|a^\dagger(k,j)a(k,j)dk.$$

Let $H_{\rm p}f=Ef$. Then

$$X_t=e^{tE}e^{-\int_0^t V(B_s+x)ds}f(B_t+x)$$

becomes a martingale for each $x\in\mathbb{R}^d$, i.e., $\mathbb{E}[X_t\mid\mathcal{F}_s]=X_s$ for the natural filtration of Brownian motion. Hence $X_{t\wedge\tau}$ is also a martingale for any stopping time τ, and

$$f(x)=\mathbb{E}[X_0]=\mathbb{E}[X_{t\wedge\tau}]=\mathbb{E}[X_\tau]$$

can be derived. From this equality we can estimate $f(x)$ by choosing an appropriate stopping time depending on V. This was done in [4] for bound states for the *relativistic* Schrödinger operator. To study (4.1) and (4.2) a similar martingale argument can be applied.

Acknowledgements: We thank Prof Hirofumi Sakuma for the kind hospitality in the conference *Basic mathematical studies on dressed photon phenomena* held in 13 - 14 of February 2020 in Institute of Math-for-Industry of Kyushu university. This work is financially supported by JSPS KAKENHI 16H03942 and CREST JPMJCR14D6.

References

[1] V. Bach, J. Fröhlich, and I.M. Sigal. Quantum electrodynamics of confined nonrelativistic particles. *Adv.Math.*, 137:299–395, 1998.

[2] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae (2nd).* Birkhäuser, 2002.

[3] R. Carmona. Pointwise bounds for Schrödinger eigenstates. *Commun. Math. Phys.*, 62:97–106, 1978.

[4] R. Carmona, W. C. Masters, and B. Simon. Relativistic Schrödinger operators: Asymptotic behavior of the eigenfunctions. *J. Funct. Anal.*, 91:117–142, 1990.

[5] C. Gérard. On the existence of ground states for massless Pauli-Fierz Hamiltonians. *Ann. H. Poincaré*, 1:443–459, 2000.

[6] C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki. Removal of UV cutoff for the Nelson model with variable coefficients. *Lett. Math. Phys.*, 101:305–322, 2012.

[7] M. Gubinelli, F. Hiroshima, and J. Lörinczi. Ultraviolet renormalization of the Nelson Hamiltonian through functional integration. *J. Funct. Anal.*, 267:3125–3153, 2014.

[8] T. Hidaka and F. Hiroshima. Pauli-Fierz model with Kato class potential and exponential decay. *Rev. Math. Phys.*, 22:1181–1208, 2010.

[9] M. Hirokawa, F. Hiroshima, and J. Lőrinczi. Spin-boson model through a Poisson driven stochastic process. *Math. Zeitschrift*, 277:1165–1198, 2014.

[10] M. Hirokawa, F. Hiroshima, and H. Spohn. Ground state for point particles interacting through a massless scalar bose field. *Adv. Math.*, 191:339–392, 2005.

[11] F. Hiroshima. Functional integral approach to semi-relativistic Pauli-Fierz models. *Adv. Math.*, 259:784–840, 2014.

[12] F. Hiroshima. *Ground states in quantum field models.* Springer, 2019.

[13] F. Hiroshima. Point-wise exponential decay of bound states of the Nelson model with Kato-class potentials. *ANALYSIS AND OPERATOR THEORY - In Honor of Tosio Kato's 100 th Birthday,Springer*, pages 225–250, 2019.

[14] F. Hiroshima. Spatial localization of bound states of models in quantum field theory. *preprint*, 2020.

[15] F. Hiroshima and J. Lörinczi. *Feynman-Kac type theorems and its applications, volume 2 (2nd ed).* De Gruyter, 2020.

[16] F. Hiroshima and O. Matte. Ground states and their associated Gibbs measures in the renormalized Nelson model. *arXiv:1903.12024*, preprint, 2019.

[17] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.*, 5:1990–1997, 1964.

[18] E. Nelson. Schrödinger particles interacting with a quantized scalar field. In *Proc. of a conference on analysis in function space, W. T. Martin and I. Segal (eds.)*, page 87. MIT Press, 1964.

[19] H. Spohn. Ground state of quantum particle coupled to a scalar boson field. *Lett. Math. Phys.*, 44:9–16, 1998.